

# Interior error estimate for periodic homogenization

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## Abstract.

In a previous article about the homogenization of the classical problem of diffusion in a bounded domain with sufficiently smooth boundary we proved that the error is of order  $\varepsilon^{1/2}$ . Now, for an open set  $\Omega$  with sufficiently smooth boundary ( $\mathcal{C}^{1,1}$ ) and homogeneous Dirichlet or Neuman limits conditions we show that in any open set strongly included in  $\Omega$  the error is of order  $\varepsilon$ . If the open set  $\Omega \subset \mathbb{R}^n$  is of polygonal ( $n=2$ ) or polyhedral ( $n=3$ ) boundary we also give the global and interior error estimates.

**Résumé.** Nous avons démontré dans un précédent article sur l'homogénéisation du problème type de la diffusion dans un domaine borné de frontière régulière que l'erreur est d'ordre  $\varepsilon^{1/2}$ . On montre maintenant pour un ouvert  $\Omega$  de frontière régulière ( $\mathcal{C}^{1,1}$ ) avec les conditions aux limites homogènes de Dirichlet ou de Neumann que dans tout ouvert fortement inclus dans  $\Omega$  l'erreur est de l'ordre de  $\varepsilon$ . Si l'ouvert  $\Omega \subset \mathbb{R}^n$  est de frontière polygonale ( $n=2$ ) ou polyédrale ( $n=3$ ) on donne également les estimations globale et intérieure de l'erreur.

**Keywords :** periodic homogenization, error estimate, unfolding method.

## 1. Introduction

This paper follows two previous studies [4,5] of the error estimates in the classical periodic homogenization problem. The first error estimates in periodic homogenization problem have been given by Bensoussan, Lions and Papanicolaou [1], by Oleinik, Shamaev and Yosifian [7], and by Cioranescu and Donato [3]. In all these works, the result is proved under the assumption that the correctors belong to  $W^{1,\infty}(Y)$ ,  $Y = ]0, 1[^n$  being the reference cell. The estimate is of order  $\varepsilon^{1/2}$ . The additional regularity of the correctors holds true when the coefficients of the operator are very regular which is not necessarily the situation in homogenization. In [4] we obtained an error estimate without any regularity hypothesis on the correctors but we supposed that the solution of the homogenized problem belonged to  $W^{2,p}(\Omega)$  ( $p > n$ ). The exponent of  $\varepsilon$  in the error estimate is inferior to  $1/2$  and depends on  $n$  and  $p$ . In [5] we obtained an error estimate without any regularity hypothesis on the correctors but we supposed that the solution of the homogenized problem belonged to  $H^2(\Omega)$ . This holds true with a smooth boundary and homogeneous Dirichlet or Neuman limits conditions. The exponent of  $\varepsilon$  in the error estimate is equal to  $1/2$ .

The aim of this work is to give the interior error estimate and new error estimate with minimal hypothesis on the boundary of  $\Omega$ .

The paper is organized as follows. Section 2 is dedicated to some projection theorems. Among them Theorems 2.3 and 2.6 are essential tools to obtain new estimates. These theorems are related to the periodic unfolding method (see [2] and [5]). We show that for any  $\phi$  in  $H^1(\Omega)$ , where  $\Omega$  is a bounded open set of  $\mathbb{R}^n$  with Lipschitz boundary, there exists a function  $\hat{\phi}_\varepsilon$  in  $L^2(\Omega; H_{per}^1(Y))$ , such that the distance between the unfolded  $\mathcal{T}_\varepsilon(\nabla_x \phi)$  and  $\nabla_x \phi + \nabla_y \hat{\phi}_\varepsilon$  is of order  $\varepsilon$  in the space  $[L^2(Y; (H^1(\Omega))')]^n$  (Theorem 2.3) and is of order  $\varepsilon^s$  in the space  $[L^2(Y; (H^s(\Omega))')]^n$ ,  $0 < s < 1$ , (Theorem 2.6), provided that the norm of gradient  $\phi$  in a neighbourhood (of thickness  $4\varepsilon\sqrt{n}$ ) of the boundary of  $\Omega$  is less than  $\varepsilon^{1/2}$  in the first case and less than  $\varepsilon^{s/2}$  in the second case.

In Theorem 3.2 in Section 3.1, we suppose that  $\Omega$  has a smooth boundary, that the right handside of the homogenization problem belongs to  $L^2(\Omega)$  and we consider the homogeneous Dirichlet or Neumann limits conditions. By transposition and thanks to Theorem 2.3 we show that the  $L^2$  error estimate is of order  $\varepsilon$  and then we obtain the interior error estimate of the same order. The required condition in Theorem 2.3 is obtained thanks to the estimates of Theorems 4.1 and 4.2 of [5].

In Theorem 3.3 in Section 3.2, we suppose that the domain  $\Omega$  is of polygonal ( $n = 2$ ) or polyhedral ( $n = 3$ ) boundary and the right handside of the homogenization problem in  $L^2(\Omega)$ . We show that the  $H^1$  error estimate is at the most of order  $\varepsilon^{1/4}$  and that the  $L^2$  and the interior error estimates are at the most of order  $\varepsilon^{1/2}$ .

We use the notation of [2] and [5] throughout this study. In this article, the constants appearing in the estimates are independent from  $\varepsilon$ .

## 2. Preliminary results

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with lipchitzian boundary. We put

$$\begin{aligned}\widehat{\Omega}_{\varepsilon,k} &= \left\{x \in \mathbb{R}^n \mid \text{dist}(x, \partial\Omega) < k\sqrt{n}\varepsilon\right\}, & \widetilde{\Omega}_{\varepsilon,k} &= \left\{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < k\sqrt{n}\varepsilon\right\}, & k &\in \{1, 2, 3, 4\}, \\ \Omega_\varepsilon &= \text{interior}\left(\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\right), & \Xi_\varepsilon &= \left\{\xi \in \mathbb{Z}^n \mid \varepsilon(\xi + Y) \cap \Omega \neq \emptyset\right\}, & Y &= ]0, 1[^n,\end{aligned}$$

where the open set  $Y = ]0, 1[^n$  is the reference cell and where  $\varepsilon$  is a strictly positive real. We have

$$\Omega \subset \Omega_\varepsilon \in \widetilde{\Omega}_{\varepsilon,1}$$

For almost any  $x \in \mathbb{R}^n$ , there exists a unique element in  $\mathbb{Z}^n$  denoted  $[x]$  such that

$$x = [x] + \{x\}, \quad \{x\} \in Y.$$

The running point of  $\Omega$  is denoted  $x$ , and the running point of  $Y$  is denoted  $y$ .

### 2.1 Projection theorems in $L^2(Y; (H^1(\Omega))')$ .

**Lemma 2.1 :** *There exists a linear and continuous extension operator  $\mathcal{P}_\varepsilon$  from  $H^1(\Omega)$  into  $H^1(\widetilde{\Omega}_{\varepsilon,3})$  such that*

$$(2.1) \quad \begin{cases} \|\nabla \mathcal{P}_\varepsilon(\phi)\|_{[L^2(\widetilde{\Omega}_{\varepsilon,3})]^n} \leq C \|\nabla \phi\|_{[L^2(\Omega)]^n} & \|\nabla \mathcal{P}_\varepsilon(\phi)\|_{[L^2(\widetilde{\Omega}_{\varepsilon,3} \setminus \Omega)]^n} \leq C \|\nabla \phi\|_{[L^2(\Omega \setminus \widehat{\Omega}_{\varepsilon,3})]^n} \\ \|\mathcal{P}_\varepsilon(\phi)\|_{L^2(\widetilde{\Omega}_{\varepsilon,3})} \leq C \left\{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla \phi\|_{[L^2(\Omega \setminus \widehat{\Omega}_{\varepsilon,3})]^n} \right\} \end{cases}$$

*The constants depend only on  $n$  and  $\partial\Omega$ .*

**Proof :** There exists a finite open covering  $(\Omega_j)_j$  of the boundary  $\partial\Omega$  such that for each  $j$  there exists a Lipschitz diffeomorphism  $\theta_j$  which maps  $\Omega_j$  to the open set  $\mathcal{O} = ]-1, 1[^{n-1} \times ]-1, 1[$  of  $\mathbb{R}^n$  and  $\Omega_j \cap \Omega$  to the open set  $\mathcal{O}_+ = ]-1, 1[^{n-1} \times ]0, 1[$ . To the covering of  $\partial\Omega$  we associate a partition of the unity

$$\phi_j \in \mathcal{C}_0^1(\Omega_j), \quad \sum_j \phi_j = 1 \quad \text{in a neighbourhood of } \partial\Omega.$$

Let  $\psi$  be in  $H^1(\Omega)$ . The function  $(\phi_j \psi) \circ \theta_j^{-1}$  belongs to  $H^1(\mathcal{O}_+)$ . We use a reflexion argument to extend this function to an element  $\widetilde{\psi}_j$  belonging to  $H^1(\mathcal{O})$ . In the neighbourhood of the boundary of  $\Omega$  the extension is equal to  $\sum_j \widetilde{\psi}_j \circ \theta_j$ . This immediately gives the estimates of Lemma 2.1.  $\square$

From now on any function belonging to  $H^1(\Omega)$  will be extended to a function belonging to  $H^1(\tilde{\Omega}_{\varepsilon,3})$ . To make the notation simpler the extension of function  $\phi$  will still be denoted  $\phi$ .

In the sequel, we will make use of definitions and results from [2] and [5] concerning the periodic unfolding method. Let us recall the definition of the unfolding operator  $\mathcal{T}_\varepsilon$  which associates a function  $\mathcal{T}_\varepsilon(\phi) \in L^1(\Omega \times Y)$  to each function  $\phi \in L^1(\Omega_\varepsilon)$ ,

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \phi\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor_Y + \varepsilon y\right) \quad \text{for } x \in \Omega \text{ and } y \in Y.$$

We also recall the approximate integration formula

$$(2.2) \quad \left| \int_\Omega v - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(v) \right| \leq \|v\|_{L^1(\hat{\Omega}_{\varepsilon,1})} \quad \forall v \in L^1(\Omega_\varepsilon)$$

For the other properties of  $\mathcal{T}_\varepsilon$ , we refer the reader to [2] and [5].

Let  $\phi \in H^1(\Omega)$  extended to  $\tilde{\Omega}_{\varepsilon,2}$ . We have defined the scale-splitting operators  $\mathcal{Q}_\varepsilon$  and  $\mathcal{R}_\varepsilon$  (see [2]). The function  $\mathcal{Q}_\varepsilon(\phi)$  is the restriction to  $\Omega$  of  $Q_1$ -interpolate of the discrete function  $M_Y^\varepsilon(\phi)$

$$M_Y^\varepsilon(\phi)(x) = \frac{1}{|Y|} \int_Y \phi\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon z\right) dz \quad x \in \Omega$$

and  $\mathcal{R}_\varepsilon(\phi) = \phi - \mathcal{Q}_\varepsilon(\phi)$ . The operator  $\mathcal{Q}_\varepsilon$  is linear and continuous from  $H^1(\Omega)$  to  $H^1(\Omega)$  and we have the estimates

$$\|\mathcal{Q}_\varepsilon(\phi)\|_{H^1(\Omega)} \leq C\|\phi\|_{H^1(\Omega)} \quad \|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n} \quad \forall \phi \in H^1(\Omega).$$

The constants depend on  $n$  and  $\partial\Omega$ .

**Theorem 2.2 :** *Let  $\phi$  be in  $H^1(\Omega)$ . There exists  $\hat{\psi}_\varepsilon$  belonging to  $H_{per}^1(Y; L^2(\Omega))$  such that*

$$(2.3) \quad \begin{cases} \|\hat{\psi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C\{\|\phi\|_{L^2(\Omega)} + \varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n}\} \\ \|\mathcal{T}_\varepsilon(\phi) - \hat{\psi}_\varepsilon\|_{H^1(Y; (H^1(\Omega))')} \leq C\varepsilon\{\|\phi\|_{L^2(\Omega)} + \varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n}\} \\ \quad + C\sqrt{\varepsilon}\{\|\phi\|_{L^2(\hat{\Omega}_{\varepsilon,2})} + \varepsilon\|\nabla\phi\|_{[L^2(\hat{\Omega}_{\varepsilon,2})]^n}\} \end{cases}$$

*The constants depend only on  $n$  and  $\partial\Omega$ .*

**Proof :** In this proof we use the same notation and the same ideas as in Proposition 3.3 of [5].

Theorem 2.2 is proved in two steps. We reintroduce the unfolding operators  $\mathcal{T}_{\varepsilon,i}$ , defined in [5], which for any  $\phi \in H^1(\Omega)$ , allow us to estimate the difference between the restrictions to two neighbouring cells of the unfolded of  $\phi$  in  $L^2(Y; (H^1(\Omega))')$ . Then we evaluate the periodic defect of the functions  $y \rightarrow \mathcal{T}_\varepsilon(\phi)(\cdot, y)$  thanks to Theorem 2.2 of [5].

Let  $K_i = Y \cup (\vec{e}_i + Y)$ ,  $i \in \{1, \dots, n\}$ . For any  $x$  in  $\Omega$ , the cell  $\varepsilon\left(\left\lfloor \frac{x}{\varepsilon} \right\rfloor_Y + K_i\right)$  is included in  $\tilde{\Omega}_{\varepsilon,2}$ .

We recall that the unfolding operator  $\mathcal{T}_{\varepsilon,i}$  from  $L^2(\tilde{\Omega}_{\varepsilon,2})$  into  $L^2(\Omega \times K_i)$  is defined by

$$\forall \psi \in L^2(\tilde{\Omega}_{\varepsilon,2}), \quad \mathcal{T}_{\varepsilon,i}(\psi)(x, y) = \psi\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor_Y + \varepsilon y\right) \quad \text{for } x \in \Omega \text{ and a. e. } y \in K_i.$$

The restriction of  $\mathcal{T}_{\varepsilon,i}(\psi)$  to  $\Omega \times Y$  is equal to the unfolded  $\mathcal{T}_\varepsilon(\psi)$  and we have the following equalities in  $L^2(\Omega \times Y)$  :

$$\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, \cdot + \vec{e}_i) = \mathcal{T}_\varepsilon(\psi)(\cdot + \varepsilon\vec{e}_i, \cdot), \quad i \in \{1, \dots, n\}$$

**Step one.** Let us take  $\psi \in L^2(\tilde{\Omega}_{\varepsilon,2})$ . We evaluate the difference  $\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, \dots + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)$  in  $L^2(Y; (H^1(\Omega))')$ . For any  $\Psi \in H^1(\Omega)$ , extended on  $\tilde{\Omega}_{\varepsilon,1}$ , a linear change of variables and the relations above give

$$\begin{aligned} \text{for a. e. } y \in Y, \quad \int_{\Omega} \mathcal{T}_{\varepsilon,i}(\psi)(x, y + \vec{e}_i) \Psi(x) dx &= \int_{\Omega} \mathcal{T}_{\varepsilon,i}(\psi)(x + \varepsilon \vec{e}_i, y) \Psi(x) dx \\ &= \int_{\Omega + \varepsilon \vec{e}_i} \mathcal{T}_{\varepsilon,i}(\psi)(x, y) \Psi(x - \varepsilon \vec{e}_i) dx \end{aligned}$$

We deduce

$$\begin{aligned} &\left| \int_{\Omega} \{ \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \} \Psi - \int_{\Omega} \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \{ \Psi(\cdot - \varepsilon \vec{e}_i) - \Psi \} \right| \\ &\leq C \| \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \|_{L^2(\hat{\Omega}_{\varepsilon,1})} \| \Psi \|_{L^2(\hat{\Omega}_{\varepsilon,1})} \quad \text{for a. e. } y \in Y. \end{aligned}$$

Since  $\Omega$  is a bounded domain with lipschitzian boundary and since  $\Psi$  belongs to  $H^1(\tilde{\Omega}_{\varepsilon,1})$  we have

$$(2.4) \quad \begin{cases} \| \Psi \|_{L^2(\hat{\Omega}_{\varepsilon,1})} \leq C \sqrt{\varepsilon} \{ \| \Psi \|_{L^2(\Omega)} + \| \nabla \Psi \|_{[L^2(\Omega)]^n} \}, \\ \| \Psi(\cdot - \varepsilon \vec{e}_i) - \Psi \|_{L^2(\Omega)} \leq C \varepsilon \left\| \frac{\partial \Psi}{\partial x_i} \right\|_{L^2(\Omega)}, \quad i \in \{1, \dots, n\}, \end{cases}$$

hence

$$\begin{aligned} &< \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y), \Psi >_{(H^1(\Omega))', H^1(\Omega)} \\ &= \int_{\Omega} \{ \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \} \Psi \\ &\leq C \varepsilon \| \nabla \Psi \|_{[L^2(\Omega)]^n} \| \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \|_{L^2(\Omega)} + C \sqrt{\varepsilon} \| \Psi \|_{H^1(\Omega)} \| \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \|_{L^2(\hat{\Omega}_{\varepsilon,1})}. \end{aligned}$$

We deduce that

$$\| \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \|_{(H^1(\Omega))'} \leq C \varepsilon \| \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \|_{L^2(\Omega)} + C \sqrt{\varepsilon} \| \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \|_{L^2(\hat{\Omega}_{\varepsilon,1})}.$$

Which leads to the following estimate of the difference between  $\mathcal{T}_{\varepsilon,i}(\psi)|_{\Omega \times Y}$  and one of its translated :

$$(2.5) \quad \| \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, \dots + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi) \|_{L^2(Y; (H^1(\Omega))')} \leq C \varepsilon \| \psi \|_{L^2(\tilde{\Omega}_{\varepsilon,3})} + C \sqrt{\varepsilon} \| \psi \|_{L^2(\hat{\Omega}_{\varepsilon,2})}.$$

The constant depends only on  $n$  and on the boundary of  $\Omega$ .

**Step two.** Let  $\phi \in H^1(\Omega)$ . The estimate (2.5) applied to  $\phi$  and its partial derivatives give us

$$\begin{aligned} &\| \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \dots + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi) \|_{L^2(Y; (H^1(\Omega))')} \leq C \varepsilon \{ \| \phi \|_{L^2(\Omega)} + \varepsilon \| \nabla \phi \|_{[L^2(\Omega)]^n} \} + C \sqrt{\varepsilon} \| \phi \|_{L^2(\hat{\Omega}_{\varepsilon,2})} \\ &\| \mathcal{T}_{\varepsilon,i}(\nabla \phi)(\cdot, \dots + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\nabla \phi) \|_{[L^2(Y; (H^1(\Omega))')^n]} \leq C \{ \varepsilon \| \nabla \phi \|_{[L^2(\Omega)]^n} + \sqrt{\varepsilon} \| \nabla \phi \|_{[L^2(\hat{\Omega}_{\varepsilon,2})]^n} \} \end{aligned}$$

We recall that  $\nabla_y(\mathcal{T}_{\varepsilon,i}(\phi)) = \varepsilon \mathcal{T}_{\varepsilon,i}(\nabla \phi)$  (see [3]). The above estimates can also be written as follows :

$$\begin{aligned} &\| \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \dots + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi) \|_{H^1(Y; (H^1(\Omega))')} \leq C \varepsilon \{ \| \phi \|_{L^2(\Omega)} + \varepsilon \| \nabla \phi \|_{[L^2(\Omega)]^n} + \sqrt{\varepsilon} \| \nabla \phi \|_{[L^2(\hat{\Omega}_{\varepsilon,2})]^n} \} \\ &\quad + C \sqrt{\varepsilon} \| \phi \|_{L^2(\hat{\Omega}_{\varepsilon,2})} \end{aligned}$$

From these inequalities, for any  $i \in \{1, \dots, n\}$ , we deduce the estimate of the difference of the traces of  $y \longrightarrow \mathcal{T}_{\varepsilon}(\phi)(\cdot, y)$  on the faces  $Y_i$  and  $\vec{e}_i + Y_i$

$$\begin{cases} \| \mathcal{T}_{\varepsilon}(\phi)(\cdot, \dots + \vec{e}_i) - \mathcal{T}_{\varepsilon}(\phi) \|_{H^{1/2}(Y_i; (H^1(\Omega))')} \leq C \varepsilon \{ \| \phi \|_{L^2(\Omega)} + \varepsilon \| \nabla \phi \|_{[L^2(\Omega)]^n} \} \\ \quad + C \sqrt{\varepsilon} \{ \| \phi \|_{L^2(\hat{\Omega}_{\varepsilon,2})} + \varepsilon \| \nabla \phi \|_{[L^2(\hat{\Omega}_{\varepsilon,2})]^n} \} \end{cases}$$

It measures the periodic defect of  $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(\cdot, y)$ . We decompose  $\mathcal{T}_\varepsilon(\phi)$  into the sum of an element belonging to  $H_{per}^1(Y; L^2(\Omega))$  and an element belonging to  $(H^1(Y; L^2(\Omega)))^\perp$  (the orthogonal of  $H_{per}^1(Y; L^2(\Omega))$  in  $H^1(Y; L^2(\Omega))$ ), see [5])

$$(2.6) \quad \mathcal{T}_\varepsilon(\phi) = \widehat{\psi}_\varepsilon + \overline{\phi}_\varepsilon, \quad \widehat{\psi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega)), \quad \overline{\phi}_\varepsilon \in (H^1(Y; L^2(\Omega)))^\perp$$

From the Riesz Theorem the dual space  $(H^1(\Omega))'$  is a Hilbert space isomorphic to  $H^1(\Omega)$ . The function  $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(\cdot, y)$  takes its values in a finite dimensionnal space,

$$\overline{\phi}_\varepsilon(\cdot, \cdot) = \sum_{\xi \in \Xi_\varepsilon} \overline{\phi}_{\varepsilon, \xi}(\cdot) \chi_\xi(\cdot)$$

where  $\chi_\xi(\cdot)$  is the characteristic function of the cell  $\varepsilon(\xi + Y)$  and where  $\overline{\phi}_{\varepsilon, \xi}(\cdot) \in (H^1(Y))^\perp$  (the orthogonal of  $H_{per}^1(Y)$  in  $H^1(Y)$ , see [5]). Hence the decomposing (2.6) is the same in  $H^1(Y; (H^1(\Omega))')$ . As the decomposing is orthogonal, we have

$$\|\widehat{\psi}_\varepsilon\|_{H^1(Y; L^2(\Omega))}^2 + \|\overline{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))}^2 = \|\mathcal{T}_\varepsilon(\phi)\|_{H^1(Y; L^2(\Omega))}^2 \leq C\{\|\phi\|_{L^2(\Omega)} + \varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n}\}^2$$

Hence we have the first inequality (2.3) and an estimate of  $\overline{\phi}_\varepsilon$  in  $H^1(Y; L^2(\Omega))$ . From Theorem 2.2 of [5] and (2.5) we obtain a finer estimate of  $\overline{\phi}_\varepsilon$  in  $H^1(Y; (H^1(\Omega))')$

$$\|\overline{\phi}_\varepsilon\|_{H^1(Y; (H^1(\Omega))')} \leq C\varepsilon\{\|\phi\|_{L^2(\Omega)} + \varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n} + \sqrt{\varepsilon}\|\nabla\phi\|_{[L^2(\widehat{\Omega}_{\varepsilon, 2})]^n}\} + C\sqrt{\varepsilon}\|\phi\|_{L^2(\widehat{\Omega}_{\varepsilon, 2})}$$

It is the second inequality in (2.3). □

**Theorem 2.3 :** For any  $\phi \in H^1(\Omega)$ , there exists  $\widehat{\phi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega))$  such that

$$(2.7) \quad \begin{cases} \|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C\|\nabla\phi\|_{[L^2(\Omega)]^n}, \\ \|\mathcal{T}_\varepsilon(\nabla_x\phi) - \nabla_x\phi - \nabla_y\widehat{\phi}_\varepsilon\|_{[L^2(Y; (H^1(\Omega))')^n]} \leq C\varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon}\|\nabla\phi\|_{[L^2(\widehat{\Omega}_{\varepsilon, 3})]^n}. \end{cases}$$

The constants depend only on  $n$  and  $\partial\Omega$ .

**Proof :** Let  $\phi \in H^1(\Omega)$ . The function  $\phi$  is decomposed

$$\phi = \Phi + \varepsilon\underline{\phi}, \quad \text{where } \Phi = \mathcal{Q}_\varepsilon(\phi) \quad \text{and} \quad \underline{\phi} = \frac{1}{\varepsilon}\mathcal{R}_\varepsilon(\phi).$$

with the following estimate :

$$(2.8) \quad \|\nabla\Phi\|_{[L^2(\Omega)]^n} + \|\underline{\phi}\|_{L^2(\Omega)} + \varepsilon\|\nabla\underline{\phi}\|_{[L^2(\Omega)]^n} \leq C\|\nabla\phi\|_{[L^2(\Omega)]^n}.$$

We apply the Poincaré-Wirtinger inequality to the function  $\phi$  in each cell of the form  $\varepsilon(\xi + K_i)$  and of the form  $\varepsilon(\xi + Y)$  included in  $\widehat{\Omega}_{\varepsilon, 3}$ . We deduce that

$$\begin{aligned} \|\nabla\mathcal{Q}_\varepsilon(\phi)\|_{[L^2(\widehat{\Omega}_{\varepsilon, 2})]^n} &\leq C\|\nabla\phi\|_{[L^2(\widehat{\Omega}_{\varepsilon, 3})]^n} \\ \implies \|\nabla\underline{\phi}\|_{[L^2(\widehat{\Omega}_{\varepsilon, 2})]^n} &\leq \frac{C}{\varepsilon}\|\nabla\phi\|_{[L^2(\widehat{\Omega}_{\varepsilon, 3})]^n} \end{aligned}$$

We also have (see [3])

$$\|\underline{\phi}\|_{L^2(\widehat{\Omega}_{\varepsilon,2})} = \frac{1}{\varepsilon} \|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\widehat{\Omega}_{\varepsilon,2})} \leq C \|\nabla \phi\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$$

Theorem 3 applied to  $\underline{\phi}$  gives us the existence of an element  $\widehat{\phi}_\varepsilon$  in  $H_{per}^1(Y; L^2(\Omega))$  such that

$$(2.9) \quad \begin{cases} \|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C \|\nabla \phi\|_{[L^2(\Omega)]^n}, \\ \|\mathcal{T}_\varepsilon(\underline{\phi}) - \widehat{\phi}_\varepsilon\|_{H^1(Y; (H^1(\Omega))')} \leq C\varepsilon \|\nabla \phi\|_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon} \|\nabla \phi\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}. \end{cases}$$

We evaluate  $\|\mathcal{T}_\varepsilon(\nabla \Phi) - \nabla \Phi\|_{[L^2(Y; (H^1(\Omega))')^n]}$ .

From Lemma 2.2 we have

$$(2.10) \quad \left\| \frac{\partial \Phi}{\partial x_i} - M_Y^\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \right\|_{(H^1(\Omega))'} \leq C\varepsilon \|\nabla \phi\|_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon} \|\nabla \phi\|_{L^2(\widehat{\Omega}_{\varepsilon,3})^n}$$

From the definition of  $\Phi$  it results that  $y \rightarrow \mathcal{T}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) (\cdot, y)$  is linear with respect to each variable. For any  $\psi \in H^1(\Omega)$ , we have

$$\begin{aligned} \langle \mathcal{T}_\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right), \psi \rangle_{(H^1(\Omega))', H^1(\Omega)} &= \int_{\Omega} \left\{ \mathcal{T}_\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) \right\} \psi \\ &= \int_{\Omega_\varepsilon} \left\{ \mathcal{T}_\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) \right\} M_Y^\varepsilon(\psi) \\ &\quad + \int_{\Omega \setminus \Omega_\varepsilon} \left\{ \mathcal{T}_\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) \right\} \psi \end{aligned}$$

We have

$$\int_{\Omega \setminus \Omega_\varepsilon} \left\{ \mathcal{T}_\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) \right\} \psi \leq C\sqrt{\varepsilon} \|\nabla \phi\|_{L^2(\widehat{\Omega}_{\varepsilon,3})^n} \{ \|\psi\|_{L^2(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)^n} \}$$

Besides, as in Theorem 3.4 of [5] we show that

$$\begin{aligned} \int_{\Omega_\varepsilon} \left\{ \mathcal{T}_\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) \right\} M_Y^\varepsilon(\psi) &\leq C\varepsilon \|\nabla \phi\|_{[L^2(\Omega)]^n} \|\nabla \psi\|_{[L^2(\Omega)]^n} \\ &\quad + C\sqrt{\varepsilon} \|\nabla \phi\|_{L^2(\widehat{\Omega}_{\varepsilon,3})^n} \{ \|\psi\|_{L^2(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)^n} \} \end{aligned}$$

and eventually

$$\forall y \in Y, \quad \left\| \mathcal{T}_\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left( \frac{\partial \Phi}{\partial x_1} \right) \right\|_{(H^1(\Omega))'} \leq C\varepsilon \|\nabla \phi\|_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon} \|\nabla \phi\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}.$$

Considering (2.10) and all the partial derivatives, we obtain

$$\|\mathcal{T}_\varepsilon(\nabla \Phi) - \nabla \Phi\|_{[L^2(Y; (H^1(\Omega))')^n]} \leq C\varepsilon \|\nabla \phi\|_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon} \|\nabla \phi\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$$

Moreover we have

$$\begin{aligned} \int_{\Omega} \frac{\partial \phi}{\partial x_i} \psi &= \int_{\partial \Omega} \underline{\phi} n_i \psi - \int_{\Omega} \underline{\phi} \frac{\partial \psi}{\partial x_i} \leq C \{ \|\underline{\phi}\|_{L^2(\partial \Omega)} + C \|\underline{\phi}\|_{L^2(\Omega)} \} \|\psi\|_{H^1(\Omega)} \\ \|\underline{\phi}\|_{L^2(\partial \Omega)} &\leq \frac{C}{\sqrt{\varepsilon}} \|\underline{\phi}\|_{L^2(\widehat{\Omega}_{\varepsilon,1})} + C\sqrt{\varepsilon} \|\nabla \underline{\phi}\|_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} \leq \frac{C}{\sqrt{\varepsilon}} \|\nabla \phi\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \end{aligned}$$

hence  $\|\varepsilon \nabla \underline{\phi}\|_{[(H^1(\Omega))']^n} \leq C\varepsilon \|\nabla \phi\|_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon} \|\nabla \phi\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$ . Thanks to (2.9) and to the above inequalities the second estimate of (2.7) is proved.  $\square$

## 2.2 Projection theorems in $L^2(Y; (H^s(\Omega))')$ , $0 < s < 1$ .

The space  $H^s(\Omega)$ ,  $0 < s < 1$ , is defined by

$$H^s(\Omega) = \left\{ \phi \in L^2(\Omega) \mid \int_{\Omega \times \Omega} \frac{|\phi(x) - \phi(x')|^2}{|x - x'|^{n+2s}} dx dx' < +\infty \right\}.$$

Equipped with the inner product

$$\langle \phi, \psi \rangle_s = \int_{\Omega} \phi \psi + \int_{\Omega \times \Omega} \frac{(\phi(x) - \phi(x'))(\psi(x) - \psi(x'))}{|x - x'|^{n+2s}} dx dx'$$

$H^s(\Omega)$  is a Hilbert separable space. We denote  $\|\cdot\|_{s,\Omega}$  the norm associated to this inner product.

As we have done in Lemma 2.1 we build a linear and continuous extension operator  $\mathcal{P}$  from  $H^s(\Omega)$ ,  $0 < s < 1$ , into  $H^s(\widetilde{\Omega}_{\varepsilon,4})$  verifying

$$\|\mathcal{P}(\phi)\|_{s,\widetilde{\Omega}_{\varepsilon,4}} \leq C \|\phi\|_{s,\Omega}$$

The constant depends only on  $n$ ,  $s$  and  $\partial\Omega$ .

From now on any function belonging to  $H^s(\Omega)$  will be extended to a function belonging to  $H^s(\widetilde{\Omega}_{\varepsilon,3})$ ,  $0 < s < 1$ . To make the notation simpler the extension of function  $\phi$  will still be denoted  $\phi$ .

**Lemma 2.4 :** *For any  $\phi \in H^s(\Omega)$ ,  $0 < s < 1$ , we have*

$$(2.11) \quad \begin{cases} \|\phi - M_Y^\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\varepsilon^s \|\phi\|_{s,\Omega} & \|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\varepsilon^s \|\phi\|_{s,\Omega} \\ \|\nabla \mathcal{Q}_\varepsilon(\phi)\|_{[L^2(\widetilde{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^{s-1} \|\phi\|_{s,\Omega}, & \|\phi\|_{L^2(\widehat{\Omega}_{\varepsilon,3})} \leq C\varepsilon^{s/2} \|\phi\|_{s,\Omega} \\ \|\phi - \phi(\cdot + \varepsilon \vec{e}_i)\|_{L^2(\Omega)} \leq C\varepsilon^s \|\phi\|_{s,\Omega}, & i \in \{1, \dots, n\} \\ \|\mathcal{Q}_\varepsilon(\phi)\|_{L^2(\partial\Omega)} \leq C\varepsilon^{(s-1)/2} \|\phi\|_{s,\Omega} \end{cases}$$

*The constants depend on  $n$ ,  $s$  and  $\partial\Omega$ .*

**Proof :** For any  $\psi$  belonging to  $H^s(Y)$ ,  $0 < s < 1$ , we have the Poincaré-Wirtinger inequality

$$\|\psi - M_Y(\psi)\|_{L^2(Y)} \leq C \|\psi\|_{s,Y}$$

where  $M_Y(\psi)$  is the mean of  $\psi$  in the cell  $Y$ . The constant depends only on  $n$ . We immediately deduce the upper bound  $\|\phi - M_Y^\varepsilon(\phi)\|_{L^2(\widetilde{\Omega}_{\varepsilon,4})} \leq C\varepsilon^s \|\phi\|_{s,\Omega}$ . We apply the Poincaré-Wirtinger inequality to the restriction of  $\phi$  to two neighbouring cells included in  $\widetilde{\Omega}_{\varepsilon,4}$  and we obtain the estimate of the gradient of  $\mathcal{Q}_\varepsilon(\phi)$  in  $\widetilde{\Omega}_{\varepsilon,3}$  ( $\|\nabla \mathcal{Q}_\varepsilon(\phi)\|_{[L^2(\widetilde{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^{s-1} \|\phi\|_{s,\Omega}$ ) and then the upper bound  $\|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\widetilde{\Omega}_{\varepsilon,3})} \leq C\varepsilon^s \|\phi\|_{s,\Omega}$  thanks to the estimate of  $\|\phi - M_Y^\varepsilon(\phi)\|_{L^2(\widetilde{\Omega}_{\varepsilon,3})}$ . The function  $\mathcal{Q}_\varepsilon(\phi)$  belongs to  $H^1(\widetilde{\Omega}_{\varepsilon,3})$ , hence considering a neighbourhood of  $\partial\widetilde{\Omega}_{\varepsilon,3}$  (included in  $\widetilde{\Omega}_{\varepsilon,3}$ ) of thickness  $\varepsilon^{1-s}$  we show that

$$\|\mathcal{Q}_\varepsilon(\phi)\|_{L^2(\widehat{\Omega}_{\varepsilon,3})} \leq C\varepsilon^{s/2} \|\phi\|_{s,\Omega} \quad \implies \quad \|\phi\|_{L^2(\widehat{\Omega}_{\varepsilon,3})} \leq C\varepsilon^{s/2} \|\phi\|_{s,\Omega}.$$

We have

$$\begin{aligned} \|\phi - \phi(\cdot + \varepsilon \vec{e}_i)\|_{L^2(\Omega)} &\leq \|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\Omega)} + \|\mathcal{Q}_\varepsilon(\phi) - \mathcal{Q}_\varepsilon(\phi)(\cdot + \varepsilon \vec{e}_i)\|_{L^2(\Omega)} \\ &\quad + \|\phi(\cdot + \varepsilon \vec{e}_i) - \mathcal{Q}_\varepsilon(\phi)(\cdot + \varepsilon \vec{e}_i)\|_{L^2(\Omega)} \leq C\varepsilon^s \|\phi\|_{s,\Omega} \end{aligned}$$

thanks to the upper bounds of  $\|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\tilde{\Omega}_{\varepsilon,3})}$  and  $\|\nabla \mathcal{Q}_\varepsilon(\phi)\|_{[L^2(\tilde{\Omega}_{\varepsilon,3})]^n}$ . The last inequality of the lemma is the consequence of the estimates  $\|\nabla \mathcal{Q}_\varepsilon(\phi)\|_{[L^2(\tilde{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^s \|\phi\|_{s,\Omega}$  and  $\|\mathcal{Q}_\varepsilon(\phi)\|_{L^2(\tilde{\Omega}_{\varepsilon,3})} \leq C\|\phi\|_{s,\Omega}$ .  $\square$

**Corollary :** For any  $s \in ]0, 1[$  and for any  $\phi \in H^s(\Omega)$  we have

$$(2.12) \quad \begin{cases} \|\mathcal{Q}_\varepsilon(\phi) - M_Y^\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\varepsilon^s \|\phi\|_{s,\Omega} \\ \|\phi - \mathcal{T}_\varepsilon(\phi)\|_{L^2(\Omega \times Y)} \leq C\varepsilon^s \|\phi\|_{s,\Omega} \end{cases}$$

The constants depend on  $n, s$  and  $\partial\Omega$ .

**Proof :** The inequalities (2.12) are the consequences of (2.11).  $\square$

**Theorem 2.5 :** Let  $\phi$  be in  $H^1(\Omega)$ . There exists  $\hat{\psi}_\varepsilon$  belonging to  $H_{per}^1(Y; L^2(\Omega))$  such that for any  $s \in ]0, 1[$

$$(2.13) \quad \begin{cases} \|\hat{\psi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C\{\|\phi\|_{L^2(\Omega)} + \varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n}\} \\ \|\mathcal{T}_\varepsilon(\phi) - \hat{\psi}_\varepsilon\|_{H^1(Y; (H^s(\Omega))')} \leq C\varepsilon^s\{\|\phi\|_{L^2(\Omega)} + \varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n}\} \\ \quad + C\varepsilon^{s/2}\{\|\phi\|_{L^2(\hat{\Omega}_{\varepsilon,2})} + \varepsilon\|\nabla\phi\|_{[L^2(\hat{\Omega}_{\varepsilon,2})]^n}\} \end{cases}$$

The constants depend only on  $n, s$  and  $\partial\Omega$ .

**Proof :** With a few modifications we prove Theorem 2.5 as Theorem 2.2. Thanks to Lemma 2.4 we replace the inequalities (2.4) of step one in Theorem 2.2 by

$$\forall \Psi \in H^s(\Omega), \quad \begin{cases} \|\Psi\|_{L^2(\hat{\Omega}_{\varepsilon,1})} \leq C\varepsilon^{s/2} \|\Psi\|_{s,\Omega}, \\ \|\Psi(\cdot - \varepsilon \vec{e}_i) - \Psi\|_{L^2(\Omega)} \leq C\varepsilon^s \|\Psi\|_{s,\Omega}, \quad i \in \{1, \dots, n\}. \end{cases}$$

$\square$

**Theorem 2.6 :** For any  $\phi \in H^1(\Omega)$ , there exists  $\hat{\phi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega))$  such that

$$(2.14) \quad \begin{cases} \|\hat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C\|\nabla\phi\|_{[L^2(\Omega)]^n}, \\ \|\mathcal{T}_\varepsilon(\nabla_x \phi) - \nabla_x \phi - \nabla_y \hat{\phi}_\varepsilon\|_{[L^2(Y; (H^s(\Omega))')^n]} \leq C\varepsilon^s \|\nabla\phi\|_{[L^2(\Omega)]^n} + C\varepsilon^{s/2} \|\nabla\phi\|_{[L^2(\hat{\Omega}_{\varepsilon,3})]^n}. \end{cases}$$

The constants depend only on  $n, s$  and  $\partial\Omega$ .

**Proof :** With a few modifications we prove Theorem 2.6 as Theorem 2.3. Proceeding as Theorem 3.4 in [5] and thanks to Lemma 2.4, we show that

$$\|\mathcal{T}_\varepsilon(\nabla\Phi) - \nabla\Phi\|_{[L^2(Y; (H^s(\Omega))')^n]} \leq C\varepsilon^s \|\nabla\phi\|_{[L^2(\Omega)]^n} + C\varepsilon^{s/2} \|\nabla\phi\|_{[L^2(\hat{\Omega}_{\varepsilon,3})]^n}$$

where  $\phi = \Phi + \varepsilon\phi$ ,  $\Phi = \mathcal{Q}_\varepsilon(\phi)$ . Now let  $\psi$  be in  $H^s(\Omega)$ . We have

$$\begin{aligned} \int_\Omega \frac{\partial \phi}{\partial x_i} \psi &= \int_\Omega \frac{\partial \phi}{\partial x_i} (\psi - \mathcal{Q}_\varepsilon(\psi)) + \int_\Omega \frac{\partial \phi}{\partial x_i} \mathcal{Q}_\varepsilon(\psi) = \int_\Omega \frac{\partial \phi}{\partial x_i} (\psi - \mathcal{Q}_\varepsilon(\psi)) + \int_{\partial\Omega} \phi n_i \mathcal{Q}_\varepsilon(\psi) - \int_\Omega \phi \frac{\partial \mathcal{Q}_\varepsilon(\psi)}{\partial x_i} \\ &\leq \|\nabla \underline{\phi}\|_{[L^2(\Omega)]^n} \|\psi - \mathcal{Q}_\varepsilon(\psi)\|_{L^2(\Omega)} + \|\underline{\phi}\|_{L^2(\partial\Omega)} \|\mathcal{Q}_\varepsilon(\psi)\|_{L^2(\partial\Omega)} + \|\underline{\phi}\|_{L^2(\Omega)} \|\mathcal{Q}_\varepsilon(\psi)\|_{[L^2(\Omega)]^n} \end{aligned}$$

hence  $\|\varepsilon \nabla \underline{\phi}\|_{[(H^s(\Omega))']^n} \leq C\varepsilon^s \|\nabla\phi\|_{[L^2(\Omega)]^n} + C\varepsilon^{s/2} \|\nabla\phi\|_{[L^2(\hat{\Omega}_{\varepsilon,3})]^n}$  thanks to the estimates of  $\underline{\phi}$  (see Theorem 2.3) and the inequalities of Lemma 2.4.  $\square$



### 3. Error estimate in the classical homogenization problem

We consider the following homogenization problem :

$$(3.1) \quad \begin{cases} \phi^\varepsilon \in H_{\Gamma_0}^1(\Omega), \\ \int_{\Omega} A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \cdot \nabla u = \int_{\Omega} f u, \\ \forall u \in H_{\Gamma_0}^1(\Omega), \end{cases}$$

where

- $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with lipschitzian boundary,
- $\Gamma_0$  is a measurable set of  $\partial\Omega$  with measure nonnull or  $\Gamma_0 = \emptyset$ ,
- $H_{\Gamma_0}^1(\Omega) = \{\phi \in H^1(\Omega) \mid \phi = 0 \text{ on } \Gamma_0\}$ ,
- $f \in L^2(\Omega)$ ,
- $A$  is a square matrix of elements belonging to  $L^\infty(Y)$ , verifying the condition of uniform ellipticity  $c|\xi|^2 \leq A(y)\xi \cdot \xi \leq C|\xi|^2$  a.e.  $y \in Y$ , with  $c$  and  $C$  strictly positive constants.

If  $\Gamma_0 = \emptyset$ , we suppose that  $\int_{\Omega} f = \int_{\Omega} \phi^\varepsilon = 0$

We have shown, see [2], that  $\nabla \phi^\varepsilon - \nabla \Phi - \mathcal{U}_\varepsilon(\nabla_y \hat{\phi})$  strongly converges towards 0 in  $[L^2(\Omega)]^n$ , where  $\mathcal{U}_\varepsilon$  is the averaging operator defined by

$$V \in L^2(\Omega \times Y) \quad \mathcal{U}_\varepsilon(V)(x) = \int_Y V\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right) dz, \quad \mathcal{U}_\varepsilon(V) \in L^2(\Omega),$$

and where

$$(\Phi, \hat{\phi}) \in H_{\Gamma_0}^1(\Omega) \times L^2(\Omega, H_{per}^1(Y)/\mathbb{R})$$

is the solution of the limit problem of unfolding homogenization

$$(3.2) \quad \begin{cases} \forall (U, \hat{u}) \in H_{\Gamma_0}^1(\Omega) \times L^2(\Omega; H_{per}^1(Y)/\mathbb{R}) \\ \int_{\Omega} \int_Y A\{\nabla_x \Phi + \nabla_y \hat{\phi}\} \cdot \{\nabla_x U + \nabla_y \hat{u}\} = \int_{\Omega} f U. \end{cases}$$

If  $\Gamma_0 = \emptyset$ , we take  $\int_{\Omega} \Phi = 0$ .

We recall that the correctors  $\chi_i$ ,  $i \in \{1, \dots, n\}$ , are the solutions of the following variational problems :

$$\chi_i \in H_{per}^1(Y), \quad \int_Y \chi_i = 0, \quad \int_Y A(y) \nabla_y (\chi_i(y) + y_i) \nabla_y \psi(y) dy = 0, \quad \forall \psi \in H_{per}^1(Y)$$

They allow us to express  $\hat{\phi}$  in terms of  $\nabla \Phi$

$$\hat{\phi} = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \chi_i,$$

and to give the homogenized problem verified by  $\Phi$

$$(3.3) \quad \int_{\Omega} A \nabla \Phi \nabla U = \int_{\Omega} f U \quad \forall U \in H_{\Gamma_0}^1(\Omega)$$

where (see [3])

$$\mathcal{A}_{ij} = \frac{1}{|Y|} \sum_{k,l=1}^n \int_Y a_{kl} \frac{\partial(y_i + \chi_i)}{\partial y_k} \frac{\partial(y_j + \chi_j)}{\partial y_l}.$$

### 3.1 First case : smooth boundary and homogeneous Dirichlet or Neumann limits conditions

In this paragraph we suppose that

- $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with  $\mathcal{C}^{1,1}$  boundary,
- $\Gamma_0 = \partial\Omega$  (homogeneous Dirichlet condition) or  $\Gamma_0 = \emptyset$  (homogeneous Neumann condition).

In Theorems 4.1 and 4.2 in [5] we gave the following error estimate for the solution of problem (3.1) :

$$(3.4) \quad \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla\phi^\varepsilon - \nabla\Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon\left(\frac{\partial\Phi}{\partial x_i}\right) \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{[L^2(\Omega)]^n} \leq C\varepsilon^{1/2} \|f\|_{L^2(\Omega)},$$

the constant depends on  $n$ ,  $A$  and  $\partial\Omega$ . In Theorem 3.2 we are going to complete these estimates.

**Lemma 3.1 :** *We have*

$$(3.5) \quad \|\nabla\phi^\varepsilon\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \leq C\sqrt{\varepsilon} \|f\|_{L^2(\Omega)}$$

The constant depends on  $n$ ,  $A$  and  $\partial\Omega$ .

**Proof :** The boundary of  $\Omega$  being of class  $\mathcal{C}^{1,1}$  we deduce that the solution  $\Phi$  of the homogenized problem (3.3.i) belongs to  $H^2(\Omega)$  and verifies  $\|\Phi\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ . The estimate of Lemma 3.1 is a consequence of (2.1), and of (3.4) and of the following inequality :

$$\begin{aligned} \|\nabla\Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon\left(\frac{\partial\Phi}{\partial x_i}\right) \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} &\leq \|\nabla\Phi\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} + C\|\nabla\mathcal{Q}_\varepsilon(\Phi)\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \|\nabla_y \chi_i\|_{[L^2(Y)]^n} \\ &\leq C\|\nabla\Phi\|_{[L^2(\widehat{\Omega}_{\varepsilon,4})]^n} \leq C\sqrt{\varepsilon} \|\Phi\|_{H^2(\Omega)} \leq C\sqrt{\varepsilon} \|f\|_{L^2(\Omega)} \end{aligned}$$

□

We denote by  $\rho(x) = \text{dist}(x, \partial\Omega)$  the distance between  $x \in \Omega$  and the boundary of  $\Omega$ .

**Theorem 3.2 :** *The solution  $\phi^\varepsilon$  of problem (3.1) verifies the following estimates :*

$$(3.6) \quad \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} \leq C\varepsilon \|f\|_{L^2(\Omega)},$$

$$(3.7) \quad \left\| \rho \left( \nabla\phi^\varepsilon - \nabla\Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon\left(\frac{\partial\Phi}{\partial x_i}\right) \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right) \right\|_{[L^2(\Omega)]^n} \leq C\varepsilon \|f\|_{L^2(\Omega)}.$$

The constants depend on  $n$ ,  $A$  and  $\partial\Omega$ .

**Proof :** We put  $\rho_\varepsilon(\cdot) = \inf\left\{\frac{\rho(\cdot)}{\varepsilon}, 1\right\}$ .

**Step one.** Let  $U \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$ . In problem (3.1) we take the test function  $U$ , then by unfolding we transform the equality we have obtained. Thanks to (2.2), (3.4) and thanks to the corollary of Proposition 3.1 of [5], we have

$$\begin{aligned} \left| \int_\Omega A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \nabla\phi^\varepsilon \cdot \nabla U - \int_{\Omega \times Y} A\mathcal{T}_\varepsilon(\nabla\phi^\varepsilon) \nabla U \right| &\leq \left| \int_\Omega A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \nabla\phi^\varepsilon \cdot \nabla U - \int_{\Omega \times Y} \mathcal{T}_\varepsilon\left(A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \nabla\phi^\varepsilon \cdot \nabla U\right) \right| \\ &\quad + \left| \int_{\Omega \times Y} A\mathcal{T}_\varepsilon(\nabla\phi^\varepsilon) \{ \mathcal{T}_\varepsilon(\nabla U) - \nabla U \} \right| \\ &\leq C\left\{ \sqrt{\varepsilon} \|\nabla\phi^\varepsilon\|_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} + \varepsilon \|\nabla\phi^\varepsilon\|_{[L^2(\Omega)]^n} \right\} \|\nabla U\|_{[H^1(\Omega)]^n} \\ &\leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n} \end{aligned}$$

We apply now Theorem 2.3 to the function  $\phi^\varepsilon$ . There exists  $\widehat{\phi}^\varepsilon \in H_{per}^1(Y; L^2(\Omega))$  such that

$$(3.8) \quad \|\mathcal{T}_\varepsilon(\nabla_x \phi^\varepsilon) - \nabla_x \phi - \nabla_y \widehat{\phi}^\varepsilon\|_{[L^2(Y; (H^1(\Omega))')]^n} \leq C\varepsilon \|f\|_{L^2(\Omega)}$$

since from Lemma 3.1 we have  $\|\nabla \phi^\varepsilon\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \leq C\sqrt{\varepsilon} \|f\|_{L^2(\Omega)}$ . From the above estimates and from (3.1) we obtain

$$(3.9) \quad \left| \int_{\Omega} f U - \int_{\Omega \times Y} A(\nabla_x \phi^\varepsilon + \nabla_y \widehat{\phi}^\varepsilon) \nabla_x U \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n}$$

Now let  $\overline{\chi}_i \in H_{per}^1(Y)$ ,  $i \in \{1, \dots, n\}$ , be the solution of the variational problem

$$(3.10) \quad \int_Y A \nabla_y \theta \nabla_y (\overline{\chi}_i + y_i) = 0 \quad \forall \theta \in H_{per}^1(Y)$$

If matrix  $A$  is symmetric  $\overline{\chi}_i = \chi_i$ ,  $\chi_i$  are the correctors.

In problem (3.1) let us take the test function  $u_\varepsilon(x) = \varepsilon \rho_\varepsilon(x) \sum_{i=1}^n \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i})(x) \overline{\chi}_i(\frac{x}{\varepsilon})$ . We have multiplied by  $\rho_\varepsilon$  so that the test function  $u_\varepsilon$  belongs to  $H_0^1(\Omega)$ . We immediately verify the inequalities ( $i \in \{1, \dots, n\}$ )

$$\begin{aligned} \left| \int_{\Omega} A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \nabla u_\varepsilon \right| &= \left| \int_{\Omega} f u_\varepsilon \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[L^2(\Omega)]^n} \\ \left| \int_{\Omega} \varepsilon A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \nabla \rho_\varepsilon \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i}) \overline{\chi}_i(\frac{\cdot}{\varepsilon}) \right| &\leq C\sqrt{\varepsilon} \|\nabla \phi^\varepsilon\|_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} \|\nabla U\|_{[H^1(\Omega)]^n} \\ \left| \int_{\Omega} \varepsilon \rho_\varepsilon A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \nabla \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i}) \overline{\chi}_i(\frac{\cdot}{\varepsilon}) \right| &\leq C\varepsilon \|\nabla \phi^\varepsilon\|_{[L^2(\Omega)]^n} \|\nabla U\|_{[H^1(\Omega)]^n} \\ \left| \int_{\Omega} (1 - \rho_\varepsilon) A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i}) \nabla_y \overline{\chi}_i(\frac{\cdot}{\varepsilon}) \right| &\leq C\sqrt{\varepsilon} \|\nabla \phi^\varepsilon\|_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} \|\nabla U\|_{[H^1(\Omega)]^n} \end{aligned}$$

From these estimates, from (3.5) and the corollary of Proposition 3.1 in [5] we obtain

$$\begin{aligned} \left| \int_{\Omega} A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i}) \nabla_y \overline{\chi}_i(\frac{\cdot}{\varepsilon}) \right| &\leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n} \\ \Rightarrow \left| \int_{\Omega} A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \sum_{i=1}^n M_Y^\varepsilon(\frac{\partial U}{\partial x_i}) \nabla_y \overline{\chi}_i(\frac{\cdot}{\varepsilon}) \right| &\leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n} \end{aligned}$$

By unfolding we transform the left handside integral of the above second inequality. From (2.2) and (3.5) we have

$$\begin{aligned} &\left| \int_{\Omega} A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \sum_{i=1}^n M_Y^\varepsilon(\frac{\partial U}{\partial x_i}) \nabla_y \overline{\chi}_i(\frac{\cdot}{\varepsilon}) - \int_{\Omega \times Y} \mathcal{T}_\varepsilon(A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \sum_{i=1}^n M_Y^\varepsilon(\frac{\partial U}{\partial x_i}) \nabla_y \overline{\chi}_i(\frac{\cdot}{\varepsilon})) \right| \\ &\leq C\sqrt{\varepsilon} \|\nabla \phi^\varepsilon\|_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} \|\nabla U\|_{[H^1(\Omega)]^n} \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n} \end{aligned}$$

We reintroduce the partial derivatives of  $U$ . As a result we have

$$\left| \int_{\Omega \times Y} A \mathcal{T}_\varepsilon(\nabla_x \phi^\varepsilon) \sum_{i=1}^n \frac{\partial U}{\partial x_i} \nabla_y \overline{\chi}_i \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n}$$

We replace  $\mathcal{T}_\varepsilon(\nabla_x \phi^\varepsilon)$  by  $\nabla_x \phi + \nabla_y \widehat{\phi}_\varepsilon$  thanks to (3.8), which gives us

$$\left| \int_{\Omega \times Y} A(\nabla_x \phi^\varepsilon + \nabla_y \widehat{\phi}_\varepsilon) \nabla_y \left( \sum_{i=1}^n \frac{\partial U}{\partial x_i} \bar{\chi}_i \right) \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n}$$

From the definition of the correctors  $\chi_i$  we obtain  $\int_{\Omega \times Y} A(\nabla_x \phi^\varepsilon + \sum_{i=1}^n \frac{\partial \phi^\varepsilon}{\partial x_i} \nabla_y \chi_i) \nabla_y \left( \sum_{j=1}^n \frac{\partial U}{\partial x_j} \bar{\chi}_j \right) = 0$ , we subtract it from the left handside of the above inequality and thanks to (3.10) we deduce

$$\left| \int_{\Omega \times Y} A \nabla_y (\widehat{\phi}_\varepsilon - \sum_{i=1}^n \frac{\partial \phi^\varepsilon}{\partial x_i} \chi_i) \nabla_x U \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n}$$

and then from (3.9) we obtain

$$(3.11) \quad \left| \int_{\Omega} \mathcal{A}(\nabla \phi^\varepsilon - \nabla \Phi) \nabla U \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[H^1(\Omega)]^n} \quad \forall U \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$$

where  $\mathcal{A}$  is the matrix of the homogenized problem.

Let  $U_\varepsilon \in H_{\Gamma_0}^1(\Omega)$  be the solution of the variationnal problem

$$(3.12) \quad \int_{\Omega} \mathcal{A} \nabla v \nabla U_\varepsilon = \int_{\Omega} (\phi^\varepsilon - \Phi) v \quad \forall v \in H_{\Gamma_0}^1(\Omega)$$

The boundary of  $\Omega$  is of class  $\mathcal{C}^{1,1}$  and we have the homogeneous Dirichlet or homogeneous Neumann limits conditions. As a result we have  $U_\varepsilon$  belonging to  $H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$ . Moreover it verifies the estimate

$$\|U_\varepsilon\|_{H^2(\Omega)} \leq C \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)}$$

In (3.12) we take  $v = \phi^\varepsilon - \Phi$  to obtain the estimate of the  $L^2$  norm of  $\phi^\varepsilon - \Phi$  thanks to (3.11).

**Step two.** Now we prove the estimate (3.7) of the theorem.

Let  $U$  be in  $H_{\Gamma_0}^1(\Omega)$ . From Theorem 3.4 in [5] there exists  $\widehat{u}_\varepsilon \in H_{per}^1(Y; L^2(\Omega))$  such that

$$(3.13) \quad \|\mathcal{T}_\varepsilon(\nabla U) - \nabla U - \nabla_y \widehat{u}_\varepsilon\|_{[L^2(Y; H^{-1}(\Omega))]^n} \leq C\varepsilon \|\nabla U\|_{[L^2(\Omega)]^n}$$

In problem (3.1) we take the test function  $\rho U$  and in problem (3.2) the couple of test functions  $(\rho U, \rho \widehat{u}_\varepsilon)$ . We obtain

$$(3.14) \quad \begin{cases} \int_{\Omega} f \rho U = \int_{\Omega} A(\{\frac{\cdot}{\varepsilon}\}) \rho \nabla \phi^\varepsilon \cdot \nabla U + \int_{\Omega} U A(\{\frac{\cdot}{\varepsilon}\}) \nabla \phi^\varepsilon \cdot \nabla \rho \\ \int_{\Omega} f \rho U = \int_{\Omega \times Y} A \rho \left( \nabla_x \Phi + \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \nabla_y \chi_i \right) (\nabla_x U + \nabla_y \widehat{u}_\varepsilon) \\ \quad + \int_{\Omega \times Y} U A \left( \nabla_x \Phi + \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \nabla_y \chi_i \right) \nabla_x \rho \end{cases}$$

The solution  $\Phi$  of homogenized problem (3.3.i) belongs to  $H^2(\Omega)$  and verifies  $\|\Phi\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$ . Hence the function  $\rho \nabla \Phi$  belongs to  $[H_0^1(\Omega)]^n$ . From (3.13) we have

$$\left| \int_{\Omega \times Y} A \rho \left( \nabla_x \Phi + \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \nabla_y \chi_i \right) (\mathcal{T}_\varepsilon(\nabla_x U) - \nabla_x U - \nabla_y \widehat{u}_\varepsilon) \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla U\|_{[L^2(\Omega)]^n}$$

Now we introduce the discrete functions  $M_Y^\varepsilon(\nabla\Phi)$ ,  $M_Y^\varepsilon(\frac{\partial\Phi}{\partial x_i})$ ,  $M_Y^\varepsilon(U)$ ,  $M_Y^\varepsilon(\rho)$ ,  $M_Y^\varepsilon(\nabla\rho)$  to replace  $\nabla\Phi$ ,  $\frac{\partial\Phi}{\partial x_i}$ ,  $U$ ,  $\rho$ ,  $\nabla\rho$  thanks to the estimate of Proposition 3.1 of [5]. We use (2.2) to transform the integrals over  $\Omega \times Y$  in integrals over  $\Omega$  by inverse unfolding. Then we replace the discrete functions by  $\nabla\Phi$ ,  $\mathcal{Q}_\varepsilon(\frac{\partial\Phi}{\partial x_i})$ ,  $U$ ,  $\rho$ ,  $\nabla\rho$  and to conclude we add the partial derivatives missing in the gradient of  $\Phi + \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon(\frac{\partial\Phi}{\partial x_i}) \chi_i(\frac{\cdot}{\varepsilon})$  (for more details see the proof of Proposition 4.3 in [5]). We obtain

$$\begin{aligned} & \left| \int_{\Omega} f \rho U - \int_{\Omega} A(\{\frac{\cdot}{\varepsilon}\}) \rho \nabla \left( \Phi + \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \nabla U \right. \\ & \quad \left. - \int_{\Omega} U A(\{\frac{\cdot}{\varepsilon}\}) \nabla \left( \Phi + \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \nabla \rho \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|U\|_{H^1(\Omega)} \end{aligned}$$

The first equality of (3.14) and the above inequality give us

$$\begin{aligned} & \left| \int_{\Omega} A(\{\frac{\cdot}{\varepsilon}\}) \rho \nabla \left( \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \nabla U \right. \\ & \quad \left. + \int_{\Omega} U A(\{\frac{\cdot}{\varepsilon}\}) \nabla \left( \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \nabla \rho \right| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|U\|_{H^1(\Omega)} \end{aligned}$$

Now we choose  $U = \rho \left( \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right)$ . From the coercivity of matrix  $A$  there follows that

$$\begin{aligned} & \|\rho \nabla \left( \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right)\|_{[L^2(\Omega)]^n}^2 \\ & \leq C \|\rho \nabla \left( \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right)\|_{[L^2(\Omega)]^n} \|\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right)\|_{L^2(\Omega)} \\ & \quad + C\varepsilon \|f\|_{L^2(\Omega)} \left\{ \|\rho \nabla \left( \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right)\|_{[L^2(\Omega)]^n} + \|\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right)\|_{L^2(\Omega)} \right\} \end{aligned}$$

Thanks to (3.6) we obtain an upper bound of  $\|\rho \nabla \left( \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right)\|_{[L^2(\Omega)]^n}$ . The functions  $\mathcal{Q}_\varepsilon(\frac{\partial\Phi}{\partial x_i})$ ,  $i \in \{1, \dots, n\}$ , are bounded in  $H^1(\Omega)$ , the estimate (3.7) immediately follows.  $\square$

**Corollary :** Let  $\Omega'$  an open set strongly included in  $\Omega$ , we have

$$\|\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right)\|_{H^1(\Omega')} \leq C\varepsilon \|f\|_{L^2(\Omega)}$$

The constant depends on  $n$ ,  $A$ ,  $\Omega'$  and  $\partial\Omega$ .  $\square$

### 3.2 Second case : Lipschitz boundary

In Theorem 4.5 of [5],  $\Gamma_0$  is a union of connected components of  $\partial\Omega$  and we have shown that there exists  $\gamma$  in the interval  $]0, 1/3]$  depending on  $A$ ,  $n$  and  $\partial\Omega$  such that the solution of problem (3.1) verifies the following error estimate :

$$(3.15) \quad \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla\phi^\varepsilon - \nabla\Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial\Phi}{\partial x_i} \right) \nabla_y \chi_i \left( \frac{\cdot}{\varepsilon} \right)\|_{[L^2(\Omega)]^n} \leq C\varepsilon^\gamma \|f\|_{L^2(\Omega)}$$

The constant depends on  $n$ ,  $A$  and  $\partial\Omega$ .

In the sequel of this paragraph we suppose that

- the open set  $\Omega$  is a bounded domain in  $\mathbb{R}^2$  of polygonal ( $n = 2$ ) or polyhedral ( $n = 3$ ) boundary,
- $\Omega$  is on one side only of its boundary,
- $\Gamma_0$  is the union of some edges ( $n = 2$ ) or some faces ( $n = 3$ ) of  $\partial\Omega$ ,
- if  $\Gamma_0 \neq \partial\Omega$  the homogenized matrix  $\mathcal{A}$  is symmetric.

We know (see [6]) that for any  $g \in L^2(\Omega)$  the solution of the variationnal problem

$$(3.16) \quad U \in H_{\Gamma_0}^1(\Omega), \quad \int_{\Omega} \nabla U \nabla \phi = \int_{\Omega} g \phi \quad \forall \phi \in H_{\Gamma_0}^1(\Omega)$$

belongs to  $H^{1+s}(\Omega)$  for an  $s$  belonging to  $]1/2, 1[$  ( $s = 1$  if the domain is convex) depending only on  $\partial\Omega$  and on the chosen limits conditions and verifies the estimate

$$\|\nabla U\|_{s,\Omega} \leq C \|g\|_{L^2(\Omega)}$$

Under a non singular linear transformation the variationnal problem (3.3) becomes (3.16). It is posed in a domain which is of the same kind as  $\Omega$ . Hence, the solution  $\Phi$  of the homogenized problem (3.3) belongs to  $H^{1+s}(\Omega)$  for an  $s$  belonging to  $]1/2, 1[$  ( $s = 1$  if the domain is convex) depending only on  $\partial\Omega$ , on  $\mathcal{A}$  and on the chosen limits conditions and verifies the estimate

$$\|\nabla \Phi\|_{s,\Omega} \leq C \|f\|_{L^2(\Omega)}$$

**Theorem 3.3 :** *The solution  $\phi^\varepsilon$  of problem (3.1) verifies*

$$(3.17) \quad \begin{cases} \left\| \nabla \phi^\varepsilon - \nabla \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right\|_{[L^2(\Omega)]^n} \leq C \varepsilon^{s/2} \|f\|_{L^2(\Omega)}, \\ \left\| \phi^\varepsilon - \Phi \right\|_{L^2(\Omega)} + \left\| \rho \left( \nabla \phi^\varepsilon - \nabla \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \right\|_{[L^2(\Omega)]^n} \leq C \varepsilon^s \|f\|_{L^2(\Omega)}. \end{cases}$$

The constants depend on  $n$ ,  $A$  and  $\partial\Omega$ .

**Proof :**

**Step one.** As in Proposition 4.3 of [5], we show that if  $(\Phi, \hat{\phi})$  is the solution of problem (3.2), then  $\Phi + \sum_{i=1}^n \varepsilon \rho_\varepsilon \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right)$  is an approximate solution of problem (3.1). The function  $\Phi$  is the solution of the homogenized problem (3.3).

Let  $\Psi \in H_{\Gamma_0}^1(\Omega)$ . Thanks to Theorem 2.6, there exists  $\hat{\psi}^\varepsilon \in H_{per}^1(Y; L^2(\Omega))$  verifying the estimates (2.15). We take  $(\Psi, \hat{\psi}^\varepsilon)$  as test-function in the unfolded problem (3.2). Since  $\nabla \Phi$  belongs to  $[H^s(\Omega)]^n$  and  $\|\nabla \Phi\|_{s,\Omega} \leq C \|f\|_{L^2(\Omega)}$ , we obtain

$$\left| \int_{\Omega} f \Psi - \frac{1}{|Y|} \int_{\Omega \times Y} A(y) \left( \nabla_x \Phi(x) + \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(x) \nabla_y \chi_i(y) \right) \mathcal{T}_\varepsilon(\nabla_x \Psi) \right| \leq C \varepsilon^{s/2} \|f\|_{L^2(\Omega)} \|\Psi\|_{H^1(\Omega)}$$

We replace  $\frac{\partial \Phi}{\partial x_i}$  by  $\mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right)$  and then, the following part of the proof is exactly the same as the proof of Proposition 4.3 in [5] because, thanks to Lemma 2.4 we have

$$(3.18) \quad \begin{cases} \|\nabla \Phi - \mathcal{Q}_\varepsilon(\nabla \Phi)\|_{[L^2(\Omega)]^n} \leq C \varepsilon^s \|f\|_{L^2(\Omega)}, \\ \|\mathcal{Q}_\varepsilon(\nabla \Phi)\|_{[L^2(\hat{\Omega}_{\varepsilon,3})]^n} \leq C \varepsilon^{s/2} \|f\|_{L^2(\Omega)}, \\ \|\mathcal{Q}_\varepsilon(\nabla \Phi)\|_{[L^2(\Omega)]^n} \leq C \|f\|_{L^2(\Omega)} \quad \|\mathcal{Q}_\varepsilon(\nabla \Phi)\|_{[H^1(\Omega)]^n} \leq C \varepsilon^{s-1} \|f\|_{L^2(\Omega)}. \end{cases}$$

Hence we obtain the first inequality of (3.17).

**Step two.** We now use the first inequality of (3.17) and again the estimates of Lemma 2.4 and as in Lemma 3.1 we prove the following upper bound of the  $L^2$  norm of gradient  $\phi^\varepsilon$  in the neighbourhood of  $\Omega$  :

$$(3.19) \quad \|\nabla \phi^\varepsilon\|_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^{s/2} \|f\|_{L^2(\Omega)}$$

The constant depends on  $n$ ,  $A$  and  $\partial\Omega$ .

**Step three.** Let  $U$  be in  $H_{\Gamma_0}^1(\Omega) \cap H^{1+s}(\Omega)$ . In problem (3.1) we take the test function  $U$ , then by unfolding we transform the equality we have obtained. Thanks to (2.2), (3.19) and thanks to the corollary of Lemma 2.4, we have

$$\begin{aligned} \left| \int_{\Omega} A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \nabla \phi^\varepsilon \cdot \nabla U - \int_{\Omega \times Y} A\mathcal{T}_\varepsilon(\nabla \phi^\varepsilon) \nabla U \right| &\leq \left| \int_{\Omega} A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \nabla \phi^\varepsilon \cdot \nabla U - \int_{\Omega \times Y} \mathcal{T}_\varepsilon\left(A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \nabla \phi^\varepsilon \cdot \nabla U\right) \right| \\ &\quad + \left| \int_{\Omega \times Y} A\mathcal{T}_\varepsilon(\nabla \phi^\varepsilon) \{\mathcal{T}_\varepsilon(\nabla U) - \nabla U\} \right| \\ &\leq C\{\varepsilon^{s/2} \|\nabla \phi^\varepsilon\|_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} + \varepsilon^s \|\nabla \phi^\varepsilon\|_{[L^2(\Omega)]^n}\} \|\nabla U\|_{s,\Omega} \\ &\leq C\varepsilon^s \|f\|_{L^2(\Omega)} \|\nabla U\|_{s,\Omega} \end{aligned}$$

We now apply Theorem 2.6 to the function  $\phi^\varepsilon$ . There exists  $\widehat{\phi}^\varepsilon \in H_{per}^1(Y; L^2(\Omega))$  such that

$$(3.20) \quad \|\mathcal{T}_\varepsilon(\nabla_x \phi^\varepsilon) - \nabla_x \phi - \nabla_y \widehat{\phi}^\varepsilon\|_{[L^2(Y; (H^s(\Omega))')^n]} \leq C\varepsilon^s \|f\|_{L^2(\Omega)}$$

We go on as in step 1 of Theorem 3.2 to obtain

$$(3.21) \quad \left| \int_{\Omega} \mathcal{A}(\nabla \phi^\varepsilon - \nabla \Phi) \nabla U \right| \leq C\varepsilon^s \|f\|_{L^2(\Omega)} \|\nabla U\|_{s,\Omega} \quad \forall U \in H_{\Gamma_0}^1(\Omega) \cap H^{1+s}(\Omega)$$

Let  $U_\varepsilon$  be the solution of the variational problem

$$(3.22) \quad U_\varepsilon \in H_{\Gamma_0}^1(\Omega), \quad \int_{\Omega} \mathcal{A} \nabla v \nabla U_\varepsilon = \int_{\Omega} (\phi^\varepsilon - \Phi) v \quad \forall v \in H_{\Gamma_0}^1(\Omega).$$

The function  $U_\varepsilon$  belongs to  $H_{\Gamma_0}^1(\Omega) \cap H^{1+s}(\Omega)$ . Moreover we have

$$\|\nabla U_\varepsilon\|_{s,\Omega} \leq C \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)}$$

We take  $v = \phi^\varepsilon - \Phi$  in (3.22) and thanks to (3.21) we obtain the estimate of the  $L^2$  norm of  $\phi^\varepsilon - \Phi$ .

**Step four.** We now prove the upper bound of  $\rho\left(\nabla \phi^\varepsilon - \nabla \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \nabla_y \chi_i\left(\frac{\cdot}{\varepsilon}\right)\right)$ .

We take a test function in  $U \in H_{\Gamma_0}^1(\Omega)$  and as in step 2 of Theorem 2.5 we decompose the unfolded of its gradient thanks to Theorem 3.4 of [5]. In (3.1) we take  $\rho U$  as test function and in (3.2) we take  $(\rho U, \rho \widehat{u}^\varepsilon)$  as couple of test functions. We obtain both equalities (3.14). In the first line of the second equality of (3.14) we replace  $\nabla \Phi$  and  $\frac{\partial \Phi}{\partial x_i}$  by  $\mathcal{Q}_\varepsilon(\nabla \Phi)$  and  $\mathcal{Q}_\varepsilon(\frac{\partial \Phi}{\partial x_i})$ . Thanks to (3.18) we have

$$\begin{aligned} &\left| \int_{\Omega} f \rho U - \int_{\Omega \times Y} A\rho\left(\mathcal{Q}_\varepsilon(\nabla_x \Phi) + \sum_{i=1}^n \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \nabla_y \chi_i\right) (\nabla_x U + \nabla_y \widehat{u}^\varepsilon) \right. \\ &\quad \left. + \int_{\Omega \times Y} U A\left(\nabla_x \Phi + \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \nabla_y \chi_i\right) \nabla_x \rho \right| \leq C\varepsilon^s \|f\|_{L^2(\Omega)} \|\nabla U\|_{[L^2(\Omega)]^n} \end{aligned}$$

From the belonging of  $\rho \mathcal{Q}_\varepsilon(\nabla \Phi)$  to  $[H_0^1(\Omega)]^n$ , and from (3.18) and from (3.13) we deduce

$$\left| \int_{\Omega \times Y} A \rho \left( \mathcal{Q}_\varepsilon(\nabla_x \Phi) + \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \right) \left( \mathcal{T}_\varepsilon(\nabla_x U) - \nabla_x U - \nabla_y \hat{u}_\varepsilon \right) \right| \leq C \varepsilon^s \|f\|_{L^2(\Omega)} \|\nabla U\|_{[L^2(\Omega)]^n}$$

We go on as in step 2 of Theorem 3.2. To conclude we use the upper bound of the  $L^2$  norm of the function  $\phi^\varepsilon - \Phi$  we obtained above.  $\square$

**Corollary :** Let  $\Omega'$  be an open set strongly included in  $\Omega$ , we have

$$\|\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right)\|_{H^1(\Omega')} \leq C \varepsilon^s \|f\|_{L^2(\Omega)}$$

The constant depends on  $n$ ,  $A$ ,  $\Omega'$  and  $\partial\Omega$ .  $\square$

**Remark :** If  $\Omega$  is a convex domain we obtain the same estimates as in Theorem 3.2.  $\square$

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